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Far Field Boundary Conditions for Unsteady Transonic Flows

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Analytical results are given for the far field of the unsteady transonic flow due to an instantaneous change in lift at the origin. These results may be superimposed to provide boundary conditions for numerical computational of unsteady transonic flows. A typical numerical experiment using these results shows substantial reductions in the size of the computational domain are possible. In two dimensions this procedure is at least as effective as the nonreflecting conditions suggested by the local characteristic relations. It is also much easier to implement in three dimensions than the nonreflecting conditions.

Introduction

THE computation of unsteady transonic flow is of fundamental importance in determining aeroelastic response and flutter boundaries.[1] At supercritical Mach numbers the flow past airfoils and wings usually includes embedded shock waves; the motion of such shock waves plays an important role in determining the airfoil or wing's response to a given mode of motion. Indeed, there is experimental evidence,[2] and supporting theoretical work,[3] suggesting that serious decrease in the flutter speed with increasing Mach number can occur for some wing designs because of this shock wave motion.

An efficient time-accurate algorithm for solving the transonic unsteady small perturbation equation has been developed by Ballhaus and Goorjian[4] for the important case of low reduced frequencies. Unsteady flows, in one sense, are easier to compute without having the results affected by approximations in the boundary conditions. One can, for example, simply insist that the boundary be far enough away that none of the waves reflected from it have sufficient time to return to the airfoil or wing and contaminate the results. For indicial and periodic motions the computational domain must be large enough that the asymptotic state is achieved before the reflected waves return to their source. This, unfortunately, turns out to be a rather large domain (typically 100 airfoil chord lengths in two dimensions). Magnus,[5] as well as other investigators, have discussed the effects of the boundary conditions on the result of their calculations, with the general conclusion being that they cause serious errors. To remedy this difficulty one may use one of several techniques. The one frequently used for steady flows, viz., grid stretching, may not improve the results, even for low reduced frequencies where the disturbance wavelength of interest is, at most, no more than 10 chord lengths. Any grid spacing larger than a few chord lengths will effectively reflect the incident waves. Another remedy, used, e.g., by Enquist and Majda,[6] is to use boundary conditions that reduce the reflection of incident waves. Such boundary conditions are a local statement that outward going waves should be transmitted through the boundary. Our experience with these boundary conditions indicates that this local approximation is much too crude to allow computational domains of size satisfactory for steady flow calculations. A better procedure is to use the global unsteady far field for the linearized equation, first attempted by Krupp and Cole.[7]

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If we assume the far field is governed by a linear equation then the far field for an indicial response can be used to construct that for any response. Additionally, we note that an effective way to proceed with flutter studies is to linearize the steady state about some experimentally or numerically determined steady state.[8] This must be done in a way that accounts for shock motions, as they represent the predominant effect in supercritical flows.[9]

Equation of Flow

For simplicity, we consider the nonlinear flow to be governed by the small perturbation for small reduced frequencies, viz.,

$$-2k\phi_{xt}+\{\kappa-(\gamma+1)\phi_x\}\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \tag{1}$$

Here the spatial coordinates, the time, and the velocity potential have been nondimensionalized by the chord, the freestream Mach number times the reciprocal of the angular frequency, and the freestream velocity times the chord, respectively; κ is the usual transonic similarity parameter, and k is the reduced frequency, viz., $k=\omega c/U$, i.e., the angular frequency multiplied by the time it takes the airfoil to move one chord length. The appropriate boundary conditions are then

$$\phi_y(x,0,z,t)=\tau[Y_x+kY_t], \quad \text{on wing}$$

and

$$[\![k\phi_t(x,0,z,t)+\phi_x(x,0,z,t)]\!]=0, \quad \text{on wake}$$

where we have included terms of $O(k)$ as suggested by Houwink and van der Vooren,[10] $\tau Y(x,y,z,t)$ is the body shape and $[\![\ ]\!]$ means the jump in the argument.

Unsteady Far Field

We now derive the unsteady far field boundary conditions for Eq. (1). Far from the body the unsteady disturbances will satisfy a linear equation. Considering the whole flowfield to be a small unsteady disturbance superimposed on the steady solution to Eq. (1), we let

$$\phi(x,y,z,t)=\phi^0(x,y,z)+\delta\psi(x,y,z,t)+o(\delta)$$

and

$$Y=Y^0+(\delta/\tau)Y^u$$

to find

$$-2k\psi_{xt}+\{[\kappa-(\gamma+1)\phi^0_x]\psi_x\}_x+\psi_{yy}+\psi_{zz}=0 \tag{2}$$

with

$$\psi_y(x,0,z,t) = (Y_x^u + kY_t^u), \quad \text{on wing}$$

where δ characterizes the size of the unsteady disturbance compared to the basic steady disturbance. While this is now the framework for a time-linearized analysis, we note that the linearized version of Eq. (1) always applies in the far field. As indicated in other studies, e.g., Klunker,[11] the nonlinear term, here correspondingly, $[(\gamma+1)\phi_x^0\psi_x]$ has a doublet like contribution to the steady far field. We will show later that such a term contributes equivalently to the unsteady far field and hence we can neglect it. Only changes in lift contribute to the lowest order; thus, we need only derive the far field for an incremental change in incidence. This we do by solving the following boundary value problem for the upper half-space ($y>0$)

$$-2\Phi_{xt} + \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \tag{3}$$

with

$$\Phi(x,0,z,t) = \tfrac{1}{2}\Delta\Gamma(z,t_0)H(t-t_0)H(x)$$

Here, the dependent and independent variables are properly scaled. The solution we seek is, of course, antisymmetric in y. The boundary condition in Eq. (3) models a vortex sheet that originates at $x=0$, producing a jump in potential, $\Delta\Gamma(z)$, instantaneously at time $t=t_0$.

Far Field Solutions

We employ the standard techniques of Fourier and Laplace transforms to solve Eq. (3) subject to the boundary condition for a change in the circulation, $\Delta\Gamma(z,t_0)$, at time t_0 for both two-dimensional and three-dimensional flow. We outline the steps briefly here.

Three Dimensions

To solve Eq. (3), we let $\hat{\Phi}(x,y,z,s)$ be the Laplace transform of Φ

$$\hat{\Phi}_{yy} + \hat{\Phi}_{xx} + \hat{\Phi}_{zz} - 2s\hat{\Phi}_x = 0 \tag{4}$$

with

$$\hat{\Phi}(x,0,z,s) = \tfrac{1}{2}\Delta\Gamma(z)(1/s)H(x)$$

The substitution $\hat{\Phi} = e^{sx}\psi$, reduces Eq. (4) to the standard Helmholtz equation, viz.,

$$\psi_{yy} + \psi_{xx} + \psi_{zz} - s^2\psi = 0 \tag{5}$$

with

$$\psi(x,0,z,s) = \tfrac{1}{2}\Delta\Gamma(z)(e^{-sx}/s)H(x)$$

We let $\hat{\psi}(\xi,y,z,s)$ denote the Fourier transform of ψ with respect to x and $\tilde{\psi}(\xi,y,\zeta,s)$ the Fourier transform of $\hat{\psi}$ with respect to z. Applying such transforms to Eq. (5), we find

$$\tilde{\psi}_{yy} - (\zeta^2 + \xi^2 + s^2)\tilde{\psi} = 0$$

with

$$\tilde{\psi}(\xi,0,\zeta,s) = \tfrac{1}{2}\Delta\tilde{\Gamma}(\zeta)\frac{s-i\xi}{s(s^2+\xi^2)}$$

which has the solution

$$\tilde{\psi}(\xi,y,\zeta,s) = \tfrac{1}{2}\Delta\tilde{\Gamma}\frac{s-i\xi}{s(s^2+\xi^2)}\exp[-(\xi^2+\zeta^2+s^2)^{1/2}y] \tag{6}$$

After performing the inverse transforms to Eq. (6) with respect to ζ,ξ,s correspondingly, we have, then, in three-space dimension, that

$$\Phi = \frac{y}{4\pi}\int_{-b/2}^{b/2}\Delta\Gamma(\bar{z},t_0)g(x,y,z-\bar{z},t-t_0)\,d\bar{z} \tag{7}$$

where b is the wing span and

$$g(x,y,z,t) = H[t+x-(x^2+y^2+z^2)^{1/2}]$$
$$\times[1+x(x^2+y^2+z^2)^{-1/2}]/(z^2+y^2)$$

Because the derivation is for an incremental change $\Delta\Gamma(z,t_0)$ at $t=t_0$, a more suitable form of Eq. (7) for Φ would be

$$\Phi(x,y,z,t) = \frac{y}{4\pi}\sum_{\Gamma(t_0)}^{\Gamma(t)}\int_{-b/2}^{b/2}\Delta\Gamma(\bar{z},t_0)g(x,y,z-\bar{z},t-t_0)\,d\bar{z} \tag{8}$$

Equation (7) is essentially the steady-state result modified by the Heaviside function which switches on the value given by Eq. (8) when

$$t-t_0 = (x^2+y^2+z^2)^{1/2} - x$$

This implies the far field phase lag is simply $(x^2+y^2+z^2)^{1/2}-x$. Such simple behavior is found only in one- and three-dimensional wave propagation (see, e.g., Lighthill[12]). Equation (8) simply retards the value of the potential Φ so that it is the potential produced by the circulation at an earlier time corresponding to the time for a wave front to travel from the origin, to the location where Φ is being evaluated. Thus, by letting

$$d\Phi_\Gamma(x,y,z,t_0) = \frac{y}{4\pi}\int_{-b/2}^{b/2}\Delta\Gamma(\bar{z},t_0)[1+x(x^2+y^2$$
$$+(z-\bar{z})^2)^{-1/2}][y^2+(z-\bar{z})^2]^{-1}d\bar{z}$$

Equation (8) becomes

$$\Phi(x,y,z,t) = \int_{t_0=0}^{t}H(t-t_0+x-\sqrt{x^2+y^2+z^2})\,d\Phi_\Gamma$$

$$= \int_0^{t-\sqrt{x^2+y^2+z^2}+x}d\Phi_\Gamma = \Phi_\Gamma[x,y,z,t-(x^2+y^2+z^2)^{1/2}+x]$$

Two Dimensions

In two dimensions we have

$$-2\Phi_{xt} + \Phi_{xx} + \Phi_{yy} = 0 \tag{9}$$

with

$$\Phi(x,0,t) = [\Delta\Gamma(t_0)/2]H(x)H(t-t_0)$$

instead of Eq. (3).

Following the same procedure used in three dimensions, we have the result in two dimensions that

$$\Phi = [\Delta\Gamma(t_0)/2\pi]f(x,y,t-t_0) \tag{10}$$

where

$$f(x,y,t) = H(t+x-\sqrt{x^2+y^2})\left[\tan^{-1}\frac{\sqrt{t^2+2xt-y^2}+t}{y}\right.$$
$$\left.+\tan^{-1}\frac{\sqrt{t^2+2xt-y^2}-t}{y}\right]$$

This result is more complex with the time appearing not only in the unit step function, but also in the argument of the arctangent function. In the limit, as we approach steady state, Eq. (10) becomes

$$f(x,y,\infty)=\frac{y}{|y|}\frac{\pi}{2}+\tan^{-1}\frac{x}{y}$$

which is the steady result of Klunker.[11] For an arbitrary circulation change $\Delta\Gamma(t)$, we must superimpose the results Eq. (10) to find the general two-dimensional far field

$$\Phi(x,y,t)=\frac{1}{2\pi}\int_0^t f(x,y,t-t_0)\frac{\mathrm{d}\Gamma(t_0)}{\mathrm{d}t_0}\mathrm{d}t_0 \qquad (11)$$

In the case of harmonic motion, e.g.,

$$\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t}=\Gamma_{\max}e^{i\omega t}$$

we simply change the lower limit in Eq. (11) to find

$$\Phi(x,y,t)/\Gamma_{\max}=\frac{e^{i\omega t}}{2\pi}\int_0^\infty f(x,y,\tau)e^{-i\omega\tau}\mathrm{d}\tau \qquad (12)$$

which gives the phase lag at each location of the far field boundary. The condition given by Krupp and Cole[7] may be viewed as an attempt to approximate the result given by Eq. (12).

We may also use Eq. (10) to examine other far field boundary conditions. For example, the one given in Ref. 6 states that for large y

$$\Phi_y\pm\Phi_x=0 \qquad (13)$$

while analytically we see that

$$\Phi_y\pm\Phi_x=-\frac{(x\mp y)(x+t)+x^2+y^2}{(x^2+y^2)(t^2+2xt-y^2)^{1/2}}\sim(x^2+y^2)^{-1/2}$$

which shows that Eq. (13), while asymptotically correct, is not a suitable replacement for Eq. (10).

Nonlinear Effect

We examine here, following one reviewer's suggestion, the nonlinear effect on the far field by adding to the right-hand side of Eq. (9) a term proportional to $(\phi_x^0\Phi_x)_x$ [see Eq. (2)]. Denoting the solution of Eq. (10) as Φ_h, we may write an integral equation for the solution Φ that

$$\Phi(x,y,t)=\Phi_h-\int_0^\infty F(x,y+\eta,\eta,t)\mathrm{d}\eta$$

$$+\int_0^y F(x,y-\eta,\eta,t)\mathrm{d}\eta+\int_y^\infty F(x,\eta-y,\eta,t)\mathrm{d}\eta \qquad (14)$$

where

$$F(x,\alpha,\eta,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(t'+x-x'-\sqrt{\alpha^2+(x-x')^2})}{\sqrt{[t'+(x-x')]^2-[\alpha^2+(x-x')^2]}}$$

$$\times\frac{\partial}{\partial x'}[\phi_{x'}^0(x',\eta)\Phi_{x'}(x',\eta,t-t')]\mathrm{d}t'\mathrm{d}x'$$

We only need to examine Eq. (14) for $y>0$, since $\Phi(x,0,t)=\Phi_h(x,0,t)$. For large x, or large α, we may approximate $F(x,\alpha,\eta,t)$ by evaluating the time integral near its singular point, i.e., $t'=-(x-x')+\sqrt{\alpha^2+(x-x')^2}$. Thus

$$F(x,\alpha,\eta,t)=\int_{-\infty}^{\infty}\ell n\sqrt{\alpha^2+(x-x')^2}\frac{\partial}{\partial x'}[\phi_{x'}^0\Phi_{x'}(x',\eta,t+x$$

$$-x'-\sqrt{\alpha^2+(x-x')^2})\mathrm{d}x'$$

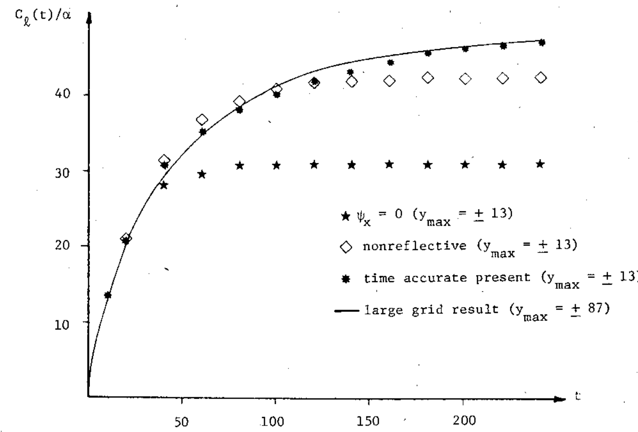

Fig. 1 Indicial pitch response for an NACA 64A006 at $M_\infty=0.88$.

or

$$F(x,\alpha,\eta,t)=\int_{-\infty}^{\infty}\frac{x-x'}{[\alpha^2+(x-x')^2]}\phi_{x'}^0\Phi_{x'}(x',\eta,t+x-x'$$

$$-\sqrt{\alpha^2+(x-x')^2})\mathrm{d}x'$$

Since both $\phi_{x'}^0$ and $\Phi_{x'}$ decay as $(x'+\eta^2)^{-1/2}$, we may further approximate $F(x,\alpha,\eta,t)$ by

$$F(x,\alpha,\eta,t)\ \frac{x}{\alpha^2+x^2}\int_{-\infty}^{\infty}\phi_{x'}^0\Phi_{x'}(x',\eta,t+x-x'$$

$$-\sqrt{\alpha^2+(x-x')^2})\mathrm{d}x' \qquad (15)$$

We recognize Eq. (15) as a doublet with strength in proportion to the compressibility effect. At large distances from the airfoil this term is negligible compared to the term we retain in Eq. (10). Thus $\phi(x,y,t)$ approaches $\phi_h(x,y,t)$ asymptotically.

Example

We illustrate the effectiveness of our results by applying them to the time development of the lift on an airfoil subjected to a step change in angle of attack. The time-linearized small perturbation algorithm of Ref. 9 is used as the test bed for the comparison.

Figure 1 compares the lift response of an NACA 64A006 airfoil at $M_\infty=0.88$, computed using different boundary conditions. Results obtained in a stretched grid with the outer boundary at a $y_{\max}$ of about 100 chord lengths away from the airfoil is compared with results obtained in a grid with a $y_{\max}$ of about 13 chord lengths.

For the large grid we find no significant difference between solutions using different boundary conditions for the times indicated in Fig. 1. In the case of the small grid, we compare solutions obtained by setting $\psi_x=0$ at $y=\pm y_{\max}$, by using the nonreflecting boundary condition $\sqrt{\kappa}\psi_x\pm\psi_y=0$ on $y=\pm y_{\max}$, and by using the result of Eq. (11) on $y=\pm y_{\max}$. The solutions are rather insensitive to upstream boundary conditions and are subject to the same downstream conditions, viz., $\psi_x=0$. The nonreflecting boundary condition of Ref. 6 achieves about 91% of the steady-state lift and gives a substantial improvement over the conventional calculation. Results from the time-accurate boundary condition we have derived here are in very good agreement with those found using the large grid; these are uncontaminated by reflections from the boundary for $t<200$.

Conclusion

We have derived the far field unsteady solutions for a step change in the lift of an airfoil and a wing. These results can be used to reduce the size of the computational domain required for either time accurate or frequency domain calculations. We have illustrated their application with a time-linearized computation of a step change in the angle of attack of a two-dimensional airfoil.

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